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FRACTIONAL REPLICATES AS ASYMMETRICAL FACTORIAL DESIGNS

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1. Introduction

Though the problem of construction of confounded symmetrical factorial designs has been solved through the efforts of Fisher, Yates, Bose, Kishen, Nair, Rao and others, much has not yet been done for confounded asymmetrical factorial designs. Yates (1937) was the first to tackle the latter type of designs and to construct a number of them. Following him, Li (1944) constructed some more of such designs. Nair and Rao (1941, 1942 and 1948) were the first to enunciate the combinatorial requisites and to suggest a number of methods of construction of such designs. Binet *et al.* (1955) developed some more designs which can be used as either single or more replicates. Recently Kishen and Srivastava (1959) have constructed such designs with the help of polynomial defined in finite geometries. But it has not yet been possible to get any single general technique for constructing such designs. An attempt has been made in this paper to supply a general technique of construction of such designs by linking them with the fractional replicates of symmetrical designs.

2. $S_1 \times S^m$ Designs in Blocks of $S_1 \times S^k$ Plots

The asymmetrical factorial design $S_1 \times S^m$ in blocks of $S_1 \times S^k$ plots can be obtained as described below through a fractional replicate of the symmetrical design (S^M, S^L) in S^L blocks with M factors each at s-levels where $L = m - k$, $M = p + m$ and p is obtained from the relation $S^{p-1} < S_1 \leqslant S^p$. The set of p factors out of M corresponding to S_1, will be called the x-pseudo factors and denoted by X_1, X_2, etc., and the other set of m factors, the real factors being denoted by A, B, C, etc.

To obtain the design (S^M, S^L) a set of $(S^L - 1)/(S - 1)$ interactions of $(S - 1)$ d.f. each will be confounded. This set of interactions will be called the confounding set of interactions and will be so chosen that no main effect or interactions of the pseudo factors is confounded. The design thus obtained will be called the parent design.

As all combinations of the x-pseudo factors are present in each of the blocks of the parent design, each block will be of the size S^{p+k} where $k \geqslant 0$. By omitting from each of the blocks those treatment combinations which contain a given number, y, of the combinations of the x-pseudo factors, we shall be left with a design with S^L blocks each of size $S^{p+k} - yS^k$, *i.e.*, $(S^p - y) S^k$.

As there are S^{m-k} blocks the total number of treatment combinations left will be $(S^p - y) \times S^m$. We shall hereafter call this set of y combinations of the x-pseudo factors the y omitted combinations.

If now y be so taken that $S^p - y = S_1$, and the remaining S_1 combinations of the x-pseudo factors be redefined to be the S_1 levels of a factor which will be called X, we shall get from the remaining portion of the present design a replicate of the asymmetrical design $S_1 \times S^m$ in blocks of $S_1 \times S^k$ plots. With the help of different confounding sets different replicates of the design can be obtained.

It will be seen that a S_1/S^p fractional replicate of the parent design has been converted into a replicate of the asymmetrical design through redefinition of the S_1 of the combinations of the pseudo-factors.

The interactions of the parent design correspond to those of the asymmetrical design through the following rule of correspondence. In future an interaction of the parent design will always mean a component of $(s - 1)$ d.f.

If any interaction of the parent design contains a set of $p' \geqslant 0$ of the pseudo factors together with some other real factors, it will correspond to that interaction of the asymmetrical design which is obtained from it by replacing the set of p'-pseudo factors in it by the single factor, X. Evidently more than one interaction of the parent design will correspond to the same interaction of the asymmetrical design unless it contains only real factors.

3. Design of the Form $S_1 \times S_2 \times S^m$ in Blocks of $S_1 \times S_2 \times S^k$ Plots

A replicate of such designs can be obtained likewise from the parent design S^M in blocks of $S^{p_1+p_2+k}$ plots, where p_1 and p_2 are obtained from $S^{p_1-1} < S_1 \leqslant S^{p_1}$, $S^{p_2-1} < S_2 \leqslant S^{p_2}$, and $M = p_1 + p_2 + m$. The p_1 of the M factors corresponding to S_1 will be called the x-pseudo factors and the p_2 factors corresponding to S_2 will be called the y-pseudo factors and will be denoted by Y_1, Y_2, etc. The other m factors in M will be called the real factors A, B, etc. Let the parent design be now obtained without confounding any main effect or interaction involving the x and y pseudo factors only. The set of interactions confounded to obtain the parent design will form the confounding set.

A replicate of the asymmetrical design will now be obtained from the parent design by first omitting from each of its block all treatment combinations containing a given set of y_1 combinations of the x-pseudo factors and another set of y_2 combinations of the y-pseudo factors such that

$$S^{p_1} - y_1 = S_1 \text{ and } S^{p_2} - y_2 = S_2$$

and then redefining the remaining combinations of the x and y pseudo factors to be the S_1 and S_2 levels of two factors which will be called respectively X and Y. The rule of correspondence between the interactions of the parent design and the asymmetrical design remains the same as before excepting that there will now be two factors X and Y each replacing its own set of pseudo factors.

The method described for obtaining a replicate of the $S_1 \times S_2 \times S^m$ design in blocks of $S_1 \times S_2 \times S^m$ plots can be generalised on the same lines for obtaining design of the type $S^n \times \Pi S_i^{m_i}$ in blocks $S^k \times \Pi S_i^{k_i}$ plots.

4. Partitioning Set and the Nature of Confounding

In order to obtain the $S.S.$ due to any main effect or interaction of the X-pseudo factors in the parent design which is required for the design $S_1 \times S^m$ the total number of S^p combinations of the x-pseudo factors is first divided into S groups of S^{p-1} each and then the $s.s.$ is obtained as usual from these group totals. If the y combinations of the x-pseudo factors which are omitted while obtaining the design $S_1 \times S^m$ contain y/S combinations from each of the above S groups, this main effect or interaction will be said to be unaffected in these

y combinations, otherwise it will be said to be confounded. The set of all main effects and interactions of the pseudo factors which are confounded in the y omitted combinations will be called the partitioning set. If y is not divisible by S, all the main effects and interactions of the x-pseudo factors will evidently form the partitioning set. If again, y is divisible by S some of the main effects and interactions may be kept unaffected by suitably choosing the y combinations. In that case the partitioning set need not contain all the main effects and interactions of the x-pseudo factors.

For getting the design $S_1 \times S_2 \times S^m$, y_1 combinations of the x-pseudo factors and y_2 combinations of the y-pseudo factors were omitted. The partitioning set will in this case consist of (*i*) all main effects and interactions of the x-pseudo factors which are confounded in the y_1 omitted combinations, (*ii*) all main effects and interactions of the y-pseudo factors which are confounded in the y_2 omitted combinations of the y-pseudo factors and (*iii*) the generalised interactions of these two sets. In this way the partitioning set can be obtained for any design.

We have so far defined two sets of affected interactions, *viz.*, (*i*) the confounding set and (*ii*) the partitioning set. We shall consider one more set of interactions which are the generalised interactions of (*i*) the partitioning set and (*ii*) the confounding set from which all interactions containing only real factors are omitted. This set of interactions will be called the generalised set. All the interactions in these three sets will be called the total confounded set. If $S_1 = S^p$ in the designs $S_1 \times S_2 \times S^m$, y_1 is zero and there will not be any main effect or interaction of the x-pseudo factors in the partitioning set.

All interactions of the parent design which correspond to those in the confounding set will be confounded. In the design $S_1 \times S_2 \times S^m$ if T_{xy} be any interaction in the confounding set containing both x and y pseudo factors, all those interactions of the asymmetrical design which correspond to the generalised interactions of T_{xy} and those in the partitioning set, will be confounded. Again, if T_x be any other interaction in the confounding set which contains x-pseudo factors but no y-pseudo factors, only those interactions of the asymmetrical design will be confounded which correspond to the generalised interactions of T_x and that portion of the partitioning set which contains only x-pseudo factors and no y-pseudo factors. This is also true for $S_1 \times S^m$ designs. Similar confounding is true for the generalised interactions T_y where T_y has similar meaning as T_x. If any interaction in the confounding set contains only real factors, it will remain completely confounded and none of its generalised interactions will be confounded.

When y is a multiple of S, the S_1 retained combinations of the X-pseudo factors in the design $S_1 \times S^m$ can be made into groups each of size S or a power of S such that in each group all the interactions of the partitioning set and none else are confounded. The main effect of the factor, X, can in this case be partitioned into two components based on between and within such groups contrasts of X.

If any interaction of the asymmetrical design involves X, it can also be partitioned into two components—one involving between group and the other within group contrasts.

If from the interactions of the confounding set which contain both real and pseudo factors, the real factors are cut out, the remainder will be interactions of the pseudo factors either belonging to the partitioning set or remaining outside it. If the remainder belongs to the partitioning set, the within group component of a confounded interaction involving X, will not be confounded. If again the remainder be outside the partitioning set, the between group component will be unconfounded.

All the main effects and interactions in the partitioning set get included in the factors X and Y, the main effects and interactions of which become free from confounding as a result of the redefinition.

Though the nature of confounding has been defined with reference to $S_1 \times S_2 \times S^m$ designs, it will be seen that generalisation is immediate.

5. Balanced Designs

The method of getting a replicate of an asymmetrical design has been described before. But analysis of single replicates of asymmetrical designs is very complicated. It is known that by including several replicates in a design, it becomes balanced and lead to simpler analysis. Thus it becomes necessary to include other replicates in the design. As a replicate can be obtained as soon as a confounding set is known, we have to search for suitable confounding sets for balance.

When a replicate of an asymmetrical design has been obtained as described earlier, two things become known, *viz.*, one is the set of interactions of the asymmetrical design which correspond to those in the confounding set and the other the total confounded set. It is now possible to get other confounding sets such that (i) each set corresponds to the same set of interactions of the asymmetrical design as above and (*ii*) each set gives rise to the same total confounded set. Any confounding set which satisfies these two conditions will be said to be a set similar to the starting confounding set or simply a similar set.

When the interactions in the starting confounding set are arranged in a row in any order excluding those containing only real factors, let E_i denote the interaction of the asymmetrical design which corresponds to the *i*-th interaction of the starting confounding set. There will actually be other interactions of the parent design which correspond to E_i. Let the set of all the interactions which correspond to E_i and are present in the total confounded set be denoted by (C_i).

Now, if n be the number of similar sets including the starting confounding set such that in these n sets (*i*) each interaction in (C_i) occurs the same number of times, say n_i, where i varies over the different interactions of the starting confounded set already arranged in a row and (*ii*) each of the interactions containing only real factors in the starting confounding set occurs in each of the n confounding sets, then the design in n replicates obtained from these n confounding sets will be said to be a balanced design.

For example, in the design 5×2^3 or 7×2^3 in 4 blocks all the seven main effects and interactions of the three x-pseudo factors, X_1, X_2 and X_3 form the partition set. Each of these designs become balanced with the following confounding sets presented in a tabular

Interactions of the asymmetrical design		XAB (E_1)	XBC (E_2)	XAC (E_3)
Starting set	(1) ..	X_1AB	X_2BC	X_1X_2AC
Similar set	(2) ..	X_2AB	X_3BC	X_2X_3AC
	(3) ..	X_3AB	X_1X_2BC	$X_1X_2X_3AC$
	(4) ..	X_1X_2AB	X_2X_3BC	X_1X_3AC
	(5) ..	X_2X_3AB	$X_1X_2X_3BC$	X_1AC
	(6) ..	$X_1X_2X_3AB$	X_1X_3BC	X_2AC
	(7) ..	X_1X_3AB	X_1BC	X_3AC

form together with the interactions of the asymmetrical design corresponding to the confounding sets.

It will be seen that each gives rise to the same total confounded set and each of the seven corresponding interactions of any of XAB, XAC and XBC in the total confounded set occurs only once in these seven sets, so that $n_i = 1$ for all the three values of i.

One more example where n_i takes different values is given below. In the design $2^2 \times 3^2$ in 9 blocks, the partitioning set consists of the main effects and interactions X_1, Y_1, X_1Y_1 and $X_1Y_1^2$. If the starting confounding set be taken as X_1Y_1A, $X_1Y_1^2B$, X_1^2AB and Y_1A^2B the design will be balanced in 4 replications obtained from the following confounding sets presented as before.

Interactions of the asymmetrical design		XYA (E_1)	XYB (E_2)	XAB (E_3)	YA^2B (E_4)
Starting set	(1) ..	X_1Y_1A	$X_1Y_1^2B$	X_1^2AB	Y_1A^2B
Similar set	(2) ..	$X_1^2Y_1^2A$	$X_1^2Y_1B$	X_1AB	$Y_1^2A^2B$
	(3) ..	$X_1Y_1^2A$	X_1Y_1B	X_1^2AB	$Y_1^2A^2B$
	(4) ..	$X_1^2Y_1A$	$X_1^2Y_1^2B$	X_1AB	Y_1A^2B

Here $n_1 = n_2 = 1$, $n_3 = n_4 = 2$

In $S_1 \times S^m$ designs where $S_1 = S^p$, all the interactions of the parent design which correspond to an interaction of the asymmetrical design are mutually independent components of $(s-1)$ $d.f.$ each of the interaction to which they correspond. Hence analysis of single replicates of such designs involves no complication. There is no partitioning set in such designs. So the starting confounding set alone will form the total confounded set. This implies that there will not be any similar set. Hence the design will be balanced in one replication. If, however, a similar set be so defined that it satisfies only the first of the two conditions in their general definition, similar sets will exist and the replicates obtained from all such similar sets will balance the design completely. The same considerations hold for designs $S_1 \times S_2 \times S^m$ when $S_1 = S^{p_1}$ and $S_2 = S^{p_2}$. In designs where $S_1 = S^{p_1}$ and $S_2 < S^{p_2}$ similar arguments hold for all the interactions in the parent design which do not contain any y-pseudo factors. If the confounding set contains an interaction without any y-pseudo factors, this interaction will be repeated just like those involving only real factors, in every similar set obtained as originally defined. The principle of balancing such designs remains the same as in the general case.

6. An Alternative Method

The method described so far holds whatever S_1, S_2, etc., may be. But when they are non-primes, alternative methods for obtaining such designs exist.

In the design $S_1 \times S^m$ let $S = rt$. If R and T be two factors with r and t levels respectively, the design $(tr) \times S^m$ in blocks of $(tr)\, S^k$ plots can be obtained from the design $r \times S^m$ in blocks of $r \times S^k$ plots by first getting from each combination of the design $r \times S^m$ t combinations by attaching to it the t levels of the factor T in turn and then taking the (tr) combinations of the factors R and T present in the different blocks to be the (tr) levels of the first factor in the design $S_1 \times S^m$. The nature of confounding as also the requirement for balance in the design $S_1 \times S^m$ remains the same as that in $r \times S^m$ when obtained through this method.

The same method can be applied to obtain the design $S_1 \times S_2 \times S^m$ in blocks of $S_1 \times S_2 \times S^k$ plots from either the design $S_1 \times S^m$ in blocks of $S_1 \times S^k$ plots or $S_2 \times S^m$ in blocks of $S_2 \times S^k$ plots by taking $S_1 = r$ and $S_2 = t$, excepting that in this case the (tr) combinations of the factors T and R are to be kept as they are and need not be redefined to form the levels of a single factor. The requirement for balance and the nature of confounding of the design $S_1 \times S_2 \times S^m$ will again remain the same as in the design from which it is obtained.

The designs obtained through this alternative method can also be obtained through the general technique by suitably choosing the starting confounding set. For example, if the confounding set does not contain any x-pseudo factor in the design $S_1 \times S_2 \times S^m$, the design obtained through the general method will be the same as that obtained from $S_2 \times S^m$ through the alternative method.

Conversely the design $q \times S^m$ in blocks of $q \times S^k$ plots where $q = S_1 \times S_2$ can be obtained from the design $S_1 \times S_2 \times S^m$ in blocks of $S_1 \times S_2 \times S^k$ plots by defining the $S_1 \times S_2$ combinations of the factors X and Y to form the q combinations of the first factor in the design $q \times s^m$.

7. Designs of the form $\Pi S_i^{m_i}$ in Blocks of $\Pi S_i^{k_i}$ Plots

We have so far described methods by means of which designs of the form $S^m \Pi S_i^{m_i}$ in blocks of $S^k \Pi S_i^{m_i}$ plots can be obtained. As in each of the blocks of such designs, all the $\Pi S_i^{m_i}$ combinations of Σm_i factors occur S^k times, each block is a multiple of a complete replication of the combinations of the Σm_i factors. Hence each such block can be divided into $S_i^{m_i - k_i}$ parts of $S^k S_i^{k_i} \prod_{j \neq i} S_j^{m_j}$ plots each through the same methods as described in the paper considering the combinations from the Σm_i factors only, and the procedure of sub-division being the same for each block. It will be possible to get ultimately the design of the form $\Pi S_i^{m_i}$ in blocks of $\Pi S_i^{k_i}$ plots through repeated subdivision of the blocks.

8. An Illustration with Analysis

The design $2 \times 2 \times 3$ in 4 plot blocks has been constructed and some details of its analysis have also been given. It has been obtained from the parent design 3^3 in 9 plot blocks by omitting all treatment combinations with the level, 2 of each of the pseudo factors X_1 and Y_1. The starting confounding set has been taken as $X_1 Y_1 A$ and hence for balance 4 replications obtainable from the four confounding sets (*i*) $X_1 Y_1 A$, (*ii*) $X_1 Y_1^2 A$, (*iii*) $X_1^2 Y_1 A$ and (*iv*) $X_1^2 Y_1^2 A$ are required. The different interactions confounded in the design are XYA, XA, YA and A. The design can also be obtained in two replications by starting with the confounding set $X_1 A$ or $Y_1 A$.

The design in 4 replications is given below:

Levels of A in different blocks

Block Nos. / XY	R_1 1	R_1 2	R_1 3	R_2 1	R_2 2	R_2 3	R_3 1	R_3 2	R_3 3	R_4 1	R_4 2	R_4 3
0 0	0	1	2	0	1	2	0	1	2	0	1	2
[illegible]	2	0	1	1	2	0	1	2	0	2	0	1
[illegible]	2	0	1	1	2	0	2	0	1	1	2	0
[illegible]	1	2	0	2	0	1	0	1	2	0	1	2

The *S.S.* of the different confounded effects can be obtained as below:

Let A_i be the *i*-th of the three totals from which the *S.S.* due to the main effect is obtained in the non-confounded case. Taking $(AX)_i$, $(AY)_i$ and $(AXY)_i$, $(i = 0, 1, 2)$ to denote similar functions of observations for obtaining the *S.S.* of the interactions AX, AY and AXY respectively in the non-confounded case, and B_{ij} as the total of the *j*-th block in the *i*-th replication, the *S.S.* due to the confounded main effect and interactions can be obtained as below:

The adjusted *S.S.* for $A = (1/240)\ \Sigma\, \text{Dev}^2 \{4A_i'\}$.

where

$$\{4A_0'\} = 4A_0 - B_{12} - B_{23} - B_{31} - B_{41}$$

$$\{4A_1'\} = 4A_1 - B_{13} - B_{21} - B_{32} - B_{42}$$

$$\{4A_2'\} = 4A_2 - B_{11} - B_{22} - B_{33} - B_{43}$$

Loss of information $= 1/16$.

The *S.S.* due to $AX = 1/208\ \Sigma\, \text{Dev}^2 \{4(AX)_i'\}$.

where

$$4(AX)_0' = 4(AX)_0 - B_{11} + B_{13} - B_{21} + B_{22} + B_{32} - B_{33} - B_{42} + B_{43},$$

$$4(AX)_1' = 4(AX)_1 + B_{11} - B_{12} - B_{22} + B_{23} - B_{31} + B_{33} + B_{41} - B_{43},$$

$$4(AX)_2' = 4(AX)_2 + B_{12} - B_{13} + B_{21} - B_{23} + B_{31} - B_{32} - B_{41} + B_{42}.$$

The *S.S.* due to $AY = 1/208\ \Sigma\, \text{Dev.}^2 \{4(AY)_i'\}$.

where

$$4(AY)_0' = 4(AY)_0 - B_{11} + B_{13} - B_{21} + B_{22} - B_{32} + B_{33} + B_{42} - B_{43},$$

$$4(AY)_1' = 4(AY)_1 + B_{11} - B_{12} - B_{22} + B_{23} + B_{31} \ldots [illegible] - B_{41} + B_{43},$$

$$4(AY)_2' = 4(AY)_2 + B_{12} - B_{13} + B_{21} - B_{23} - \ldots [illegible] + B_{41} - B_{42}.$$

Loss of information $= 3/16$ for both AX and AY.

The $S.S.$ due to $AXY = 1/112\ \Sigma\,\mathrm{Dev}^2\,\{4\,(AXY)_i'\}$,

where

$$4\,(AXY)_0' = 4\,(AXY)_0 + 3\,(B_{12} + B_{23} - B_{31} - B_{41})$$

$$4\,(AXY)_1' = 4\,(AXY)_1 + 3\,(B_{13} + B_{21} - B_{32} - B_{42})$$

$$4\,(AXY)_2' = 4\,(AXY)_2 + 3\,(B_{11} + B_{22} - B_{33} - B_{43})$$

Loss of information $= 9/16$.

9. Designs of the Type 6×2^m and 10×2^m in Two Blocks per Replication

In designs of the type 6×2^m we have seen that the main effect of the factor X as also its interaction with one or more of the real factors can be subdivided in two components which have been called the between and within group components. The method of construction of balanced design described in the paper ensures that only one of these two components will be affected and balanced while the other will remain unaffected. The confounding of the interaction is thus partial and enjoys all the merits and demerits of partial confounding in factorial designs. As against this type of confounding there are designs where confounding of interaction of any order is balanced. Dr. Yates, while going through an earlier version of the paper, expressed the view that in certain designs involving particularly only two factors, *i.e.*, in designs like 6×2 in six plot blocks it is desirable to have such balanced loss of information of all the components of the affected interaction, otherwise the estimation of any other type of component of the interaction which is physically interpretable may be difficult.

Though in designs involving two factors only there is a necessity of this type of balance over all the components of the affected interaction, it appears in designs with more than two factors where no interaction with less than three factors is affected this type of complete balance may not be so much necessary, particularly when the affected interactions can be neglected. Following a suggestion from Dr. Yates a method of construction of such completely balanced design has been obtained as presented below. The method has been developed with reference to the 6×2 design in 6 plot blocks. The steps necessary for its generalisation has been indicated afterwards. Let the two factors be termed as usual X and A where X has the level 0, 1, 2, 3, 4, and 5 and A, 0 and 1. Let the six levels of the factor X be allotted in the 6 plots of a block. Next in order to keep the main effect of A unaffected three of these plots should receive the level 0 of A, the other three plots receiving its other level 1. The other block required to complete a replication can be obtained from this block by writing the level 0 for 1 and 1 for 0 of the second factor. In this replication evidently only the two-factor interaction is affected. For complete balance of the two-factor interaction a number of other such blocks, one for each replication, are to be so chosen that the plot numbers receiving say 0 level of the second factor forms a B.I.B. design in the blocks. For balance of this design, therefore, a B.I.B. design with the parameters $v = 6$, $k = 3$, is necessary. As a B.I.B. design with the paremeter $v = 6$, $k = 3$, $b = 10$, $r = 5$ and $\lambda = 2$ is available, the 6×2 balanced design can be obtained in 10 replications with the help of the above design.

The design 10×2 in 10 plot blocks can similarly be balanced with 18 replications with the help of the B.I.B. design $v = 10$, $b = 18$, $k = 5$, $r = 9$ and $\lambda = 4$. The 6×2^2 design in blocks of 12 plots where only the three-factor interaction is affected can be balanced in 10 replications similarly with the help of the same B.I.B. design by replacing 0 and 1, levels of the factor A by $\alpha_0 \equiv (00, 11)$ and $\alpha_1 \equiv (01, 10)$ where 00, 11, 10, and 01 are the four combinations of the two levels of each of factors A and B.

10. Rationale behind the Method

In the design $S_1 \times S^m$, each of the S^p treatment combinations of the pseudo factors can be expressed as a linear function of the main effects and interactions of the pseudo factors. Next let I_R denote any real factor interaction component with $(S - 1)$ *d.f.* Now the totality of $S_1 \times S^m$ combinations obtained after omitting the y combinations of the pseudo factors can be formed into a two-way classification corresponding to a $S_1 \times S$ table where S_1 stands for the S_1 existing combinations of the pseudo factors and S for the S levels of the real factor interactions. The analysis of the variance partitioning of this table gives the components due to (*i*) S_1 combinations of the pseudo factors, *i.e.*, the S_1 levels of the real factor with $(S_1 - 1)$ *d.f.*, (*ii*) the real factor interaction with $(S - 1)$ *d.f.* and (*iii*) the generalised interaction of the real factor interaction and the combinations of the pseudo factors. We shall call this third interaction the mixed interaction which is also the interaction XI_R. Any component of this mixed interaction can always be expressed as a linear function of the effects of the interactions which are the generalised interactions of the real factor interaction and those in the partitioning set. When $S_1 < S^p$, any component of the real factor interaction also can always be expressed as a linear function of such interaction effects.

When S_1 is a multiple of S, the S_1 combinations can be made into S_1/S groups as indicated earlier so that the mixed interaction gets divided into two components, *viz.*, (i) between group mixed interaction and (ii) within group mixed interaction. In such case, the between group component can be expressed as linear function of a part of the generalised interactions and the within group component can be expressed as the function of the rest of the generalised interactions. More details about the point have been included earlier in the paper. From above it is now clear that if any of the generalised interactions be confounded, both the mixed and the real factor interaction will be affected.

If again the starting confounding set contains any one of these generalised interactions of the real factor interaction (it cannot contain more than one of these interactions) no information on this interaction can be obtained. As such all these components of the mixed interaction which contain this interaction cannot also be estimated. If now another replicate be obtained through another confounding set containing some other generalised interaction, it will be possible to estimate all the components of the mixed interaction, but the different components will be estimated with different precisions if there be more than two generalised interactions. This argument shows that if each of the components of the mixed interaction, *i.e.*, the interaction XI_R is to be estimated with equal precision, each of these generalised interactions must be confounded the same number of times, as has been indicated earlier for obtaining balanced designs.

In the design $S_1 \times S_2 \times S^m$ we can form a 3-way table in a similar way with (i) S_1 combinations of the x-pseudo factors, (ii) S_2 combination of the y-pseudo factors and (iii) S levels of the real factor interaction. In this case there will be three mixed interactions, *viz.*, (i) without X, (ii) without Y, and (iii) with both X and Y. Any component of the mixed interaction without Y can be expressed as a linear function of the effects of those generalised interactions of the real factor interaction and those in the partitioning set which do not contain an Y-factor. Hence when any of the generalised interactions without any Y-factor is confounded, only the mixed interaction without Y-factor will be confounded. Through similar arrangements the nature of confounding of the other two mixed interactions follow.

The considerations for balance remain the same as described in $S_1 \times S^m$ designs.

As an example in the design 5×2^2 in 10 plot blocks, the form of the two-way table is as below:

Pseudo factorial combination	Level of interaction AB	
	$(AB)_0$	$(AB)_1$
000	00000,00011	00001,00010
001	00100,00111	00101,00110
010	01000,01011	01001,01010
011	01100,01111	01101,01110
100	10000,10011	10001,10010

The function, 00000 + 00011 + 00101 + 00110 − 00001 − 00010 − 00100 − 00111 is a component of the mixed interaction. When expressed as a function of the effects of interaction and main effects the contrast becomes:

$$4(x_3ab)_0 - 4(x_3ab)_1 + 4(x_2x_3ab)_0 - 4(x_2x_3ab)_1 + 4(x_1x_3ab)_0 - 4(x_1x_3ab)_0 + 4(x_1x_2x_3ab)_0 - 4(x_1x_2x_3ab)_1.$$

Thus if anyone of the interactions with X_3 be confounded, to obtain a replication, this component cannot be estimated from it.

Again the contrast 01100 + 01111 + 10001 + 10010 − (10000 + 10011 + 01101 + 01110) is estimable from

$$4(x_1ab)_0 - 4(x_1ab)_1 + 4(x_2ab)_1 - 4(x_2ab)_0 + 4(x_3ab)_1 - 4(x_3ab)_0 + 4(x_1x_2x_3ab)_0 - 4(x_1x_2x_3ab)_1.$$

If now two replications be taken confounding, say, X_1AB and X_2AB, the second contrast will have less precision than the first one none of the interactions in which is affected. If again, X_3AB and $X_1X_2X_3AB$ be confounded in two replications both these contrasts will be estimated with equal precision, but there will be other contrasts which cannot be estimated with that precision. This shows that unless each of the generalised interactions be confounded an equal number of times, the different components of the mixed interaction cannot be estimated with the same precision. It has been stated earlier that the interaction AB is also expressible as a function of the generalised interactions together with its own effects. In the present case, the AB interacted contrast involves the function:

$$3\,(x_1ab)_0 + (x_2ab)_0 + (x_3ab)_0 + (x_1x_2ab)_1 + (x_1x_3ab)_1 + (x_2x_3ab)_0 + (x_1x_2x_3ab)_1.$$

Thus, for the estimation of AB independently of the mixed interaction the assumption that the above function is zero, is necessary. If any one or more of the interactions be confounded, the block adjusted interaction AB will no longer involve the same function of the interaction effects, unless each of the generalised interactions is confounded the same number of times. Thus for independence also the same number of replications is necessary.

SUMMARY

A method of constructing confounded asymmetrical factorial designs from fractional replicates of symmetrical designs through redefinition of factors has been described. Nature of confounding as also conditions for balance have also been investigated. The method has been illustrated by means of examples.

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